Then, for $t \geqslant t_1 = \xi_1\varepsilon^{-1}$ we have condition (11), and for $t \geqslant t_2 = \varepsilon^{-1} \geqslant t_1$, the function $s(x, t) > 0,\ 0 < x \leqslant 1$.

Scheme of the proof. We introduce the function v by the equation

$$\int_0^t a(\tau)\, d\tau = \exp(\beta t) v + \varphi, \qquad \beta = \text{const}$$

In the domain $G_+ = \{1 - \varepsilon t \leqslant x \leqslant 1, 0 \leqslant t \leqslant t_2\}$ it will satisfy an equation and boundary condition from which, since $\varphi, s \geqslant 0$ and in G_+: $\varphi_{xx} = \varphi_{\xi\xi}$, $\varphi_x = \varphi_\xi$, $\varphi_t = \varepsilon\varphi_\xi$ (the subscript ξ means differentiation with respect to ξ) in the case of large β, we find by the maximum principle that $v \geqslant 0$ in G_+ Hence the theorem follows.

Corollary 3. Using Theorem 4 and the dissertation mentioned near the start, we can show that, for small t, the quantity $s(1, t)$ increases as $F_0^{-1}(t)$.

An example of relative permeability and capillary pressure functions which satisfy the conditions of Theorems 1-4 is given by

$$f_1(s) = (1 - s)(1 - s + {}^1/_3 s^2),\ f_2(s) = \mu^\circ(1 - f_1) + (1 - \mu^\circ)s^2(3 - 2s)$$
$$p_c(s) = [s/(1,1 - s)]^{1/2},\ \mu^\circ = \mu_2/\mu_1 \leqslant 1$$

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THE SOLUTION OF PROBLEMS OF ELASTICITY THEORY BY THE METHOD OF ANALYTIC FUNCTIONS*

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Sherman's method /3/ is used to investigate some two-dimensional problems of elasticity theory for multiply connected regions. The solution is constructed by series expansion of the Kolosov-Muskhelishvili potential. Using Faber polynomials and conformal mapping, the original problem is reduced to the solution of linear systems of infinite algebraic equations in the expansion coefficients of analytical functions. The solution procedure is demonstrated by two examples.

Many problems of elasticity theory reduce /1-3/ to finding analytic functions of the complex variable $z = x + iy$ that are regular (or sinlge-valued) in a given region S and satisfy appropriate initial conditions.

For the plane problem (the first boundary-value problem), when S is bounded by several smooth closed contours $L_1, L_2, \ldots, L_k$ such that the last contour encloses all the previous

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contours and they have no common points, the boundary condition on each contour L_k $(k = 1, 2, \ldots)$ has the form

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f_1^{(k)} + i f_2^{(k)} \text{ on } L_k \tag{0.1}$$

$$f_1^{(k)} + i f_2^{(k)} = i \int_{t_0}^{t} [T_{nx}^{(\,)} + i T_{ny}^{(k)}]\, ds + C_k$$

For torsion and bending problems,

$$\varphi(t) + \overline{\varphi(t)} = f(t) + C_k \text{ on } L_k \tag{0.2}$$

$$f(t) = \begin{cases} t\bar{t} + C_k & \text{for torsion problems} \\ {}^1/_3 (2+\mu) y^3 - \mu x^3 y + 4(1+\mu) \int & \text{for bending problems} \end{cases}$$

Here t identifies the points on the contours L_k, t_0 is an arbitrary fixed point on L_k, and $T_{nx}^{(k)}$ and $T_{ny}^{(k)}$ are the projections of the external forces on the axes x and y, respectively.

If a uniformly distributed load p_k is applied to the boundary of the region, then on each contour we have the equality

$$f_1^{(k)} + i f_2^{(k)} = -p_k t + C_k$$

Here C_k are some constants, one of which may be fixed arbitrarily.

The accuracy of the solution of problem (0.1) and (0.2) largely depends on the correct choice of the analytical function and the mapping function (mapping the region S onto the unit circle).

1. Selection of mapping functions. The exterior of a regular curvilinear (nearly linear) polygon is mapped on the exterior of the unit circle in the plane ξ_2 by the function

$$z = A\left(\xi_2 + \frac{m}{\xi_2^{q-1}}\right); \quad A = \frac{a+b}{2}, \quad m = \left|\frac{a-b}{a+b}\right| \tag{1.1}$$

where a is the radius of the circle described around the polygon, b is the radius of the circle inscribed inside the polygon, and q is the number of symmetry axes (the number of sides).

The inverse of function (1.1) is defined by the equality

$$\xi_2 = \chi_2(z) = u \sum_{n=0}^{E(k/q)} a_n^{(1)} u^{-qn}, \quad u = \frac{z}{A} \tag{1.2}$$

where $E(k/q)$ is the integer part of k/q.

Retaining the first six terms in the series (1.2), we obtain: for $q = 6$ (a regular hexogon)

$$\chi_2(z) = u(1 - mu^{-6} - 5m^2u^{-12} - 40m^3u^{-18} - 385m^4u^{-24} - 4095m^5u^{-30} \ldots) \tag{1.3}$$

for $q = 4$ (a square)

$$\chi_2(z) = u(1 - mu^{-4} - 3m^2u^{-8} - 15m^3u^{-12} - 91m^4u^{-16} - 612m^5u^{-20} \ldots) \tag{1.4}$$

for $q = 2$ (an ellipse with semiaxes a and b)

$$\chi_2(z) = u(1 - mu^{-2} - 2m^2u^{-4} - 5m^3u^{-6} - 14m^4u^{-8} - 42m^5u^{-10} + \ldots) \tag{1.5}$$

In all expansions (1.3)-(1.5), $|z| \geqslant A$.

The exterior of the circle of radius r with two straight cuts along the abscissa axis is mapped to the exterior of the unit circle in the plane ξ_1 by the function

$$z = r \sum_{n=-1}^{\infty} \gamma_n \xi_1^{-n} = r\xi_1 \sum_{n=0}^{\infty} \gamma_{n-1} \xi_1^{-n}. \tag{1.6}$$

The inverse of function (1.6) is given by

$$\xi_1 = \frac{z}{r} \sum_{n=0}^{E(n/2)} \delta_{n-1}^{(1)} \left(\frac{r}{z}\right)^{2n}$$

$$\delta_{n-1}^{(1)} = \sum_{k=0}^{n}{}^{*} 2\lambda (-1)^{k/2} C_{1/2}^{k/2} \left(\frac{1}{2\alpha}\right)^{-k+1} C_{-k+1}^{(n-k)/2}, \quad \lambda = \begin{cases} 1, & k = 0 \\ 1/2, & k \neq 0 \end{cases}$$

The first 10 coefficients in expansion (1.6) were found:

$$\gamma_1 = \alpha = \frac{e^2 + r^2}{2er} > 1, \quad \gamma_1 = \alpha - \frac{1}{\alpha}, \quad \gamma_3 = \frac{1}{\alpha} - \frac{1}{\alpha^3}$$
$$\gamma_5 = -\frac{1}{\alpha} + \frac{3}{\alpha^3} - \frac{2}{\alpha^5}, \quad \gamma_7 = \frac{1}{\alpha} - \frac{6}{\alpha^3} + \frac{10}{\alpha^5} - \frac{5}{\alpha^7}$$
$$\gamma_9 = -\frac{1}{\alpha} + \frac{10}{\alpha^3} - \frac{30}{\alpha^5} + \frac{35}{\alpha^7} - \frac{14}{\alpha^9}, \quad \gamma_0 = \gamma_2 = \gamma_4 = \gamma_6 = \gamma_8 = \gamma_{10} = 0$$

($\pm e$ is the coordinate of the endpoint of the cut).

If the inner contour L_1 in our problem consists of a circle of radius r with two straight cuts of different lengths, then the exterior of this contour is mapped on the exterior of the unit circle in the plane ξ_1 by the function

$$z = r\left(w + \sqrt{w^2 - 1}\right), \quad w = \frac{b' - a'}{4}\left(\xi_1 + \frac{1}{\xi_1}\right) + \frac{b' + a'}{2}, \quad -\infty < a' < b' < \infty.$$

After some transformations, this formula can be simplified to the form (which is convenient for inversion)

$$z = r\xi_1 \sum_{n=0}^{\infty} \lambda_n \xi_1^{-n}, \quad \lambda_n = \sum_{\nu=n-2E(n/2)}^{n} \delta_\nu \gamma_{n-\nu} \tag{1.7}$$
$$\gamma_k = \sum_{\nu=k-2E(k/2)}^{k/2} \varepsilon (-1)^\nu C_{1/2}^{\nu} \left(\frac{b' - a'}{4}\right)^{-2\nu+1} C_{-2\nu+1}^{(k-2\nu)/2}, \quad \varepsilon = \begin{cases} 2, & \nu = 0 \\ 1, & \nu \neq 0 \end{cases}$$
$$\delta_n = \sum_{n_1=n-2E(n/2)}^{n}{}^{*} C_{-2\nu+1}^{n_1} \left(2\,\frac{b' + a'}{b' - a'}\right)^{n_1} C_{-n}^{(n-n_1)/2}$$
$$a' = \frac{e_1^2 + r^2}{2e_1 r}, \quad b' = \frac{e_2^2 + r^2}{2e_2 r}$$

Here e_1 and e_2 are the endpoint coordinates of the cut.

In particular, for $e_1 = -r$ (then $a' = -1$) or for $e_2 = r$ (then $b' = 1$), the inner contour L_1 has one cut. For $|e_1| = |e_2| = e$ we have $a' = -\alpha$, $b' = \alpha$. Then formulas (1.7) and (1.6) are identical (in this case, $\lambda_n = \gamma_n$, $\delta_0 = 1$, $\delta_n = 0$ /4-6/).

The inverse of function (1.7) is given by

$$\xi_1 = \frac{z}{r} \sum_{n=0}^{\infty} \varepsilon_n^{(1)} \left(\frac{r}{z}\right)^n, \quad \varepsilon_n^{(1)} = \sum_{k=n-2E(n/2)}^{n}{}^{*} h_k^{(1)} g_{n-k} \tag{1.8}$$
$$g_\nu = \sum_{n=\nu-2E(\nu/2)}^{\nu/2} \varepsilon (-1)^n C_{1/2}^{n} (b' - a')^{2n-1} C_{-2n+1}^{(\nu-2n)/2}$$
$$h_\nu^{(1)} = \sum_{k=\nu-2E(\nu/2)}^{\nu}{}^{*} (-1)^k C_{-2n+1}^{k} (b' + a')^k \, C_{-k}^{(\nu-k)/2}$$

2. ***Selection of analytical (regular) functions.*** The function $\varphi(z)$ regular in the biconnected region S can be represented as the sum of two functions, one of which $(f_1(z))$ is regular inside the outer contour L_2 and the other $(f_2(z))$ is regular outside the inner contour L_1. Thus, $\varphi(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \sum_{k=1}^{\infty} \alpha_k \xi_1^{-k}, \quad f_2(t) = \sum_{k=-\infty}^{\infty} \beta_k [\chi_2(t)]^k \tag{2.1}$$

Multiplying both sides of the second equality in (2.1) by the Cauchy kernel $[2\pi i(t - z)]^{-1} dt$ and integrating over the entire contour L_2, we obtain by residue theory, using expansion (1.2).

$$f_2(z) = \beta_0 + \sum_{k=1}^{\infty} \beta_k \left(\frac{z}{A}\right)^k \sum_{n=0}^{E(k/q)} a_n^{(k)} \left(\frac{A}{z}\right)^{qn} \tag{2.2}$$

Using equalities (1.8) and (2.2), we rewrite (2.1) in the form

$$\varphi(z) = \sum_{k=0}^{\infty} a_k \left(\frac{r}{z}\right)^k + \sum_{k=0}^{\infty} b_k \left(\frac{z}{A}\right)^k \tag{2.3}$$

$$a_k = \sum_{\nu=k-E(k/2)}^{k}{}^{*} \alpha_k \delta_{-1}^{-k} l_{(k-\nu)/2}^{(\nu)}, \quad b_k = \sum_{\nu=k}^{\infty}{}^{**} \beta_\nu a_{(\nu-k)/q}^{(\nu)}$$

The asterisks ** and * indicate that the summation index is incremented, respectively, by 2 and by q at each step.

Note that if one of the contours L_1, L_2 is a circle, then some of the formulas are simplified, but equality (2.3) remains unchanged.

The function $\psi(z)$ has the form (similar to expansion (2.3))

$$\psi(z) = \sum_{\nu=0}^{\infty} d_\nu \left(\frac{r}{z}\right)^\nu + \sum_{\nu=0}^{\infty} h_\nu \left(\frac{z}{A}\right)^\nu \tag{2.4}$$

$$d_\nu = \sum_{k=\nu-E(\nu/2)}^{\nu}{}^{**} A_k \delta_{-1}^{-k} l_{(\nu-k)/2}^{(k)}, \quad h_\nu = \sum_{k=\nu-E(\nu/q)}^{\nu}{}^{*} B_k a_{(\nu-k)/q}^{(k)}$$

Substituting the expressions for the analytical functions defined by equalities (2.3) and (2.4) into (0.1), we rewrite the boundary conditions on L_j $(j = 1, 2)$ for the plane problem in the form

$$\sum_{k=0}^{\infty} d_k \left(\frac{r}{t}\right)^k + \sum_{k=0}^{\infty} b_k \left(\frac{t}{A}\right)^k - t \sum_{k=0}^{\infty} a_k \frac{r^k}{(\bar{t})^{k+1}} + t \sum_{k=0}^{\infty} b_k k t^{k-1} A^{-k} +$$
$$\sum_{k=0}^{\infty} d_k \left(\frac{r}{t}\right)^k + \sum_{k=0}^{\infty} h_k \left(\frac{\bar{t}}{A}\right)^k = f_1^{(j)} + i f_2^{(j)} \quad \text{on} \, L_j \tag{2.5}$$

Similarly, for torsion and bending problems,

$$\sum_{k=1}^{\infty} a_k \left(\frac{r}{t}\right)^k + \sum_{k=0}^{\infty} b_k \left(\frac{t}{A}\right)^k + \sum_{k=0}^{\infty} d_k \left(\frac{r}{\bar{t}}\right)^k + \sum_{k=0}^{\infty} b_k \left(\frac{\bar{t}}{A}\right)^k = f(t) + C_j \quad \text{on } L_j \tag{2.6}$$

3. ***Examples.*** *The stressed state of a hexagonal plate with a central circular cavity and two straight cuts (Fig.1).* A homogeneous isotropic plate is subjected to a uniformly distributed load around the contour. using the mapping functions (1.1) and (1.6) in (2.5), we reduce the boundary conditions on L_j by simple transformations to four linear systems of algebraic equations and take $q = 6$ and $\tau\bar{\tau} = 1$.

In numerical calculations, the first five terms were retained in each system. The calculations were carried out for the following ratios of the main dimensions of the plate cross-section: version 1: $r/A = 0.5$, $e/A = 0.6$, $b/r = 1.92$; version 2 $r/A = 0.7$, $e/A = 0.8$, $b/r = 1.371428$.

The stresses σ_r and σ_θ calculated at the characteristic points of the cross-section relative to the uniform load P applied to the outer contour (the load on the inner contour was zero) are presented in Table 1.

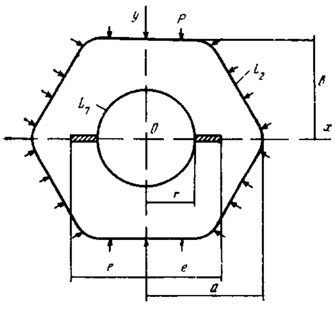

Fig.1

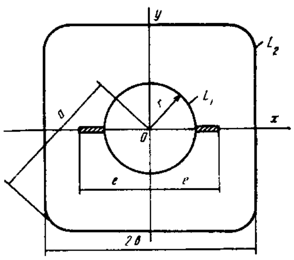

Fig.2

At the endpoints of the cut $z = \pm e$, we determined the stress intensity factor (SIF) and the critical value of the contour load. The SIF values expressed in units of $10K/(P\sqrt{l})$ (where $l = e - r$ is the cut length) were 6.72 for version 1 and 12.84 for version 2.

For the critical contour load (when crack propagation begins) we obtained $P_* = 1.534\sigma_b$ for version 1 and $P_* = 0.803\,\sigma_b$ for version 2, where σ_b is the average technical strength of the material.

If we take $A = R$ and $m = 0$ in the linear systems of infinite algebraic equations described above, the problem reduces to finding the stressed state of a circular plate weakened by a central circular hole and two straight cuts. For $r/R \leqslant 0.2$ and $l \leqslant r$ the results obtained for σ_θ, SIF, and the critical load P agree with previously published data /7/.

The torsion of a hollow square beam with two cuts (Fig.2). From the boundary conditions (2.6), using the mapping functions (1.1) and (1.6), we obtain after some transformations (equating the coefficients of equal powers of the variable τ) two linear systems of infinite algebraic equations.

Table 1

Version	z/r	σ_r/P	$-\sigma_\theta/P$
1	1,21	0,0012	2,68
	1,3	0,44	2,22
	1.6	—0,25	1.98
	2,0	—1.0014	1,72
2	1,15	0,003	4,21
	1.2	—0.12	3,84
	1.3	0,34	3,33
	1,43	—1,002	2,43

Table 2

Version	r/b	$\tau_{xz}/(\mu\tau b)$	$\tau_{yz}/(\mu\tau b)$
1	0,7	—	1,769
	0,8	—	0,84
	1,0	—	1,3617
	l	1.354	—
2	0.8	—	1.96
	0,9	—	0,972
	1,0	—	1,47
	l	1.362	—

Retaining the first five equations in each system, we find the shear stresses τ_{xz} and τ_{yz} from the formula

$$\tau_{xz} - i\tau_{yz} = \mu\tau b\,[\varphi'(z) - \bar{z}]$$

where τ is the torsion angle; for version 1: $r/b = 0.5$, $e/b = 0.6$; for version 2. $r/b = 0.5$, $e/b = 0.7$.

The values of τ_{xz} and τ_{yz} calculated at characteristic points of the beam cross-section are given in Table 2.

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